Received March 1998
Revised January 1999
Accepted February 1999

Free convection flow of thermomicropolar fluid along a vertical plate with nonuniform surface temperature and surface heat flux

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Keywords *Laminar free convection, Micropolar fluids, Heat transfer, Boundary layers*

Abstract *We determine the effects of micro-inertia density and the vortex viscosity on laminar free convection boundary layer flow of a thermomicropolar fluid past a vertical plate with exponentially varying surface temperature as well as surface heat flux. The governing nonsimilarity boundary layer equations are analyzed using: first, a series solution for small ξ (a scaled streamwise distribution of micro-inertia density), second, an asymptotic solution for large ξ and, third, a full numerical solution implicit finite difference method together with Keller-box scheme. Results are expressed in terms of local skin friction and local Nusselt number. The effects of varying the vortex viscosity parameter, Δ, surface temperature and the surface heat flux gradient n and m respectively against ξ for fluids having Prandtl number equals 0.72 and 7.0 are determined.*

Nomenclature

f = dimensionless stream function
g = dimensionless microrotation
g = acceleration due to gravity
Gr = Grashof number
h = heat transfer coefficient
j = microinertia per unit mass
k = thermal conductivity of the fluid
N = angular velocity
Nu = Nusselt number
Pr = Prandtl number
q_w = surface heat flux
T = temperature
u, v = velocity components
x, y = distance along and normal to the surface
ψ = stream function
ν = viscosity coefficient
ρ = density of the fluid
κ = rotational viscosity coefficient
β = volumetric coefficient of expansion
γ = gyroviscosity coefficient
ξ = local microinertia density parameter
η = dimensionless normal coordinate
θ = dimensionless temperature
Δ = dimensionless material property

1. Introduction

Buoyancy forces that arise from density differences in a fluid cause free convection. These density differences are a consequence of temperature

International Journal of Numerical Methods for Heat & Fluid Flow, Vol. 9 No. 5, 1999, pp. 568-585.

gradients within the fluid. Free convection flow is a significant factor in several practical applications that include, for example, cooling of electronic components.

There exist relatively few studies concerning the non-Newtonian fluids with microstructures such as polymeric additives, colloidal suspensions, animal blood, liquid crystals etc. Eringen (1966) developed the theory of micropolar fluids, which shows microrotation effects as well as microinertia. The theory of thermomicropolar fluids was developed by Eringen (1972) by extending the theory of micropolar fluids. Gorla (1983) investigated the forced convective heat transfer to a micropolar fluid flow over a flat plate. Jena and Mathur (1981), by means of similar method, studied the free convective heat transfer to a thermomicropolar fluid along a vertical flat plate. Recently, Hossain *et al.* (1995) and Hossain and Chowdhury (1995) investigated the effect of material parameters on the mixed convection flow of thermomicropolar fluid from a vertical as well as a horizontal heat surface, taking into consideration that the spin-gradient viscosity is non-uniform. In these analyses, they obtained appropriate transformations to demonstrate the flow in the entire mixed convection regime and the results of the resulting equations were obtained using the implicit finite difference method.

In this paper, we have studied the problem of natural convection boundary layer flow of a micropolar fluid over a vertical plate with non-uniform surface temperature or heat flux conditions. The numerical results revealed the presence of a two-layer structure as the distance from the leading edge increases. The existence of an inner layer close to the wall is due to the restriction imposed by the wall on the rotation of the microelements in a fluid, as explained by Rees and Bassom (1966). An asymptotic analysis for large distances away from the leading edge is presented since accurate numerical results are difficult to obtain in this region because of the near-wall regime. Numerical results for the friction factor and Nusselt number are presented for different values of the material parameters of the fluid.

2. Mathematical formalisms

A two-dimensional steady free convection flow of a viscous incompressible thermomicropolar fluid along a non-isothermal vertical flat impermeable plate is considered. The temperature of the ambient fluid is assumed to be uniform at T_∞ while the temperature at the surface of the plate is considered to be proportional to x^n and the surface heat flux to x^m (x measures the distance from the leading edge along the surface of the plate and m, n are dimensionless temperature gradient parameters). The flow configuration and the coordinate system are shown in Figure 1.

Under the usual Boussinesq approximation, the flow is governed by the following boundary layer equations (Jena and Mathur, 1951):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

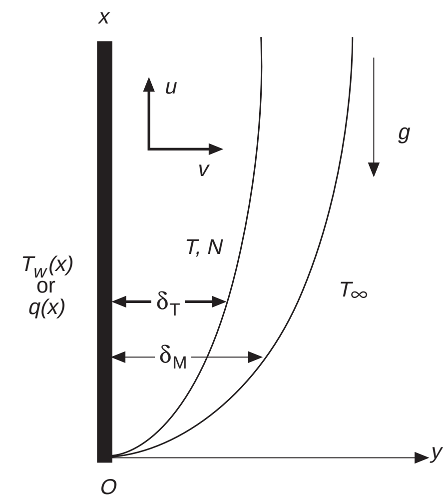

Figure 1.
The coordinate system and the flow configuration

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=(\mu+\kappa)\frac{\partial^2 u}{\partial y^2}+\kappa\frac{\partial N}{\partial y}+\rho \boldsymbol{g}\beta(T-T_\infty) \tag{2}$$

$$\rho j\left(u\frac{\partial N}{\partial x}+v\frac{\partial N}{\partial y}\right)=\gamma\frac{\partial^2 N}{\partial y^2}-\kappa\left(2N+\frac{\partial u}{\partial y}\right) \tag{3}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial r}=\alpha\frac{\partial^2 T}{\partial y^2} \tag{4}$$

where, u, v are respectively the x and y-components of the velocity field, ν is the kinematic coefficient of viscosity, T is the temperature of the fluid in the boundary layer, p is pressure, ρ is the density of the fluid, $\boldsymbol{g}$ is the acceleration due to gravity, β is the coefficient of volume expansion, and α is the thermal diffusivity, C_p is the specific heat at constant pressure, N is the microrotation component normal to (x, y)-plane, j is the micro-inertia density, κ is the vortex viscosity, γ is the spin-gradient viscosity given by $\gamma = (\mu + \kappa/2)j$ (see Rees and Bassom, 1966). We follow the works of many recent authors by assuming the microinertia density, j, as constant.

The boundary conditions to be satisfied are

$$\begin{gathered} y=0: \;\; u=v=0, \;\; T=T_w(x) \;\; \text{and} \;\; k\left(\frac{\partial T}{\partial y}\right)=-q(x), \;\; N=-s\frac{\partial u}{\partial y} \\ y\to\infty: \;\; u\to 0, \;\; \zeta=0, \;\; T\to T_\infty \end{gathered} \tag{5}$$

In equation (5), we have followed Rees and Bassom (1966) by assigning a variable relation between N and the surface shear-stress; where s is the microrotation parameter. The value $s = 0$ corresponds to the case where particle density is sufficiently great that microelements close to the wall are unable to rotate. The value $s = 1/2$ is indicative of weak concentrations and when $s = 1$, we have flows which are representative of turbulent boundary layer (see Peddieson and McNitt). Throughout the present investigation we shall consider value of $s = 1/2$. Further, in equation (5), the conditions at the surface of the plate, i.e. at $y = 0$, the former one is for the nonisothermal plate and the latter for the non-uniform heat flux.

Prescribed surface temperature

For the nonisothermal plate, we introduce the following variables

$$\psi = \nu Gr_x^{1/4} f(\xi,\eta), \quad N = \frac{\nu}{x^2}\mathrm{Gr}_x^{3/4} g(\xi,\eta), \; \theta = \frac{\mathrm{T}-\mathrm{T}_\infty}{T_w - T_\infty}, \quad T_w - T_\infty = T_0 x^n$$
$$\eta = \frac{y}{x} Gr_x^{1/4}, \quad \xi = \frac{x^2}{j} Gr_x^{-1/2} \tag{6}$$

where η is the pseudosimilarity variable, ξ is the local nonsimilarity variable that measures the streamwise distribution of the combined effects of the microinertia density and the buoyancy force defined thereby, and ψ is the stream function such that $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, T_0 is a constant, n is the temperature gradient parameter, such that

$$n = \frac{d\ln(T_w - T_\infty)}{d\ln x} \tag{7}$$

Introducing the transformation given in equation (6) into the set of equations (2)-(4) one gets

$$(1+\Delta)f''' + \frac{n+3}{4}ff'' - \frac{n+1}{2}f'^2 + \theta + \Delta g' = \frac{1}{2}(1-n)\xi\left(f'\frac{\partial f'}{\partial\xi} - f''\frac{\partial f}{\partial\xi}\right) \tag{8}$$

$$(1+\tfrac{1}{2}\Delta)g'' + \frac{n+3}{4}fg' - \frac{3n+1}{4}f'g - \Delta\xi(2g+f'') = \frac{1}{2}(1-n)\xi\left(f'\frac{\partial g}{\partial\xi} - g'\frac{\partial f}{\partial\xi}\right) \tag{9}$$

$$\frac{1}{\mathrm{Pr}}\theta'' + \frac{n+3}{4}f\theta' - nf'\theta = \frac{1}{2}(1-n)\xi\left(f'\frac{\partial\theta}{\partial\xi} - \theta'\frac{\partial f}{\partial\xi}\right) \tag{10}$$

where

$$\Pr = \frac{\nu}{\alpha}, Gr_x = \frac{g\beta(T_w - T_\infty)}{\nu^2} x^3 and \Delta = \frac{\kappa}{\mu} \tag{11}$$

represent, respectively, the Prandtl number, the local Grashof number for the nonisothermal plate and spin-gradient viscosity.

The corresponding boundary conditions given in (5) take the form

$$\begin{gathered} f(0,\xi) = f'(0,\xi) = 0,\ \theta(0,\xi) = 1,\ g(0,\xi) = -\tfrac{1}{2} f''(0,\xi) \\ f'(\infty,\xi) = 0,\ \theta(\infty,\xi) = g(\infty,\xi) = 0 \end{gathered} \tag{12}$$

where primes denote differentiation of the functions with respect to η.

At this stage, it is worthwhile to draw attention to the case for which equations (8) - (10) are satisfied by similarity solutions; this is when $\xi = 0$.

The solutions of equations (8)-(10) together with boundary conditions (12) enable us to calculate various flow parameters, such as the local skin-friction, C_f, the local Nusselt number, Nu_x, at the surface and the distribution of local couple-stress, M_x, in the boundary layer from the following relations:

$$\tfrac{1}{2} Gr_x^{1/4} C_f = \left[1 + \frac{1}{2}\Delta\right] f''(\xi, 0) \tag{13}$$

$$Gr_x^{-1/4} Nu_x = \theta'(\xi, 0) \tag{14}$$

and

$$M_x(\xi,\eta) = \frac{\gamma(\partial N/\partial y)}{\rho g \beta (T_w - T_\infty) j} = (1 + \Delta/2) g'(\xi,\eta) \tag{15}$$

Prescribed surface heat flux

For the free convection flow along a vertical plate with non-uniform surface heat flux, the following variables may be introduced

$$\begin{gathered} \psi = \nu Gr_x^{1/5} f(\xi,\eta),\quad N = \frac{\nu}{x^2} Gr_x^{3/5} g(\xi,\eta),\quad T - T_\infty = \frac{qx}{k} Gr_x^{-1/5} \theta(\xi,\eta), \\ q = q_0 x^m\ \eta = \frac{y}{x} Gr_x^{1/5},\ \xi = \frac{x^2}{j} Gr_x^{-2/5},\ Gr_x = \frac{g\beta q}{k\nu^2} x^4 \end{gathered} \tag{16}$$

where, q_0 and m are constants, η is the pseudosimilarity variable and ξ is the local nonsimilarity variable that measures the streamwise distribution of the combined effects of the microinertia density and the buoyancy force for the present case. Introducing the above transformations, given in equation (16), into the set of equations (2)-(4) we get

$$(1+\Delta)f''' + \frac{m+4}{5}ff'' - \frac{2m+3}{5}f'^2 + \theta + \Delta g' = \frac{2}{5}(1-m)\xi\left(f'\frac{\partial f'}{\partial \xi} - f''\frac{\partial f}{\partial \xi}\right) \tag{17}$$

$$\begin{aligned}(1+\Delta/2)g'' + \frac{m+4}{5}fg' - \frac{3m+2}{5}f'g - \Delta\xi(2g+f'')\\ = \frac{2}{5}(1-m)\xi\left(f'\frac{\partial g}{\partial \xi} - g'\frac{\partial f}{\partial \xi}\right)\end{aligned} \tag{18}$$

$$\frac{1}{\Pr}\theta'' + \frac{m+4}{5}f\theta' - \frac{1-m}{5}f'\theta = \frac{2}{5}(1-m)\xi\left(f'\frac{\partial \theta}{\partial \xi} - \theta'\frac{\partial f}{\partial \xi}\right) \tag{19}$$

The corresponding boundary conditions given in (5) take the form

$$\begin{aligned} f(0,\xi) = f'(0,\xi) = 0,\ \theta'(0,\xi) = -1,\ g(0,\xi) = -\tfrac{1}{2}f''(0,\xi)\\ f'(\infty,\xi) = 0,\ \theta(\infty,\xi) = g(\infty,\xi) = 0\end{aligned} \tag{20}$$

As before, here also, the primes denote differentiation of the functions with respect to η.

In this case, the various flow parameters, such as the local skin-friction, C_f, the local Nusselt number, Nu_x, at the surface and the distribution of local couple-stress, M_x, in the boundary layer may be calculated from the following relations:

$$Gr_x^{-3/5}C_f = \left[1 + \frac{1}{2}\Delta\right]f''(\xi,0) \tag{21}$$

$$Gr_x^{-1/5}Nu_x = 1/\theta(\xi,0) \tag{22}$$

and

$$M_x(\xi,\eta) = \frac{\gamma(\partial N/\partial y)}{\rho \mathbf{g}\beta(T_w - T_\infty)j} = \left(1 + \tfrac{1}{2}\Delta\right)g'(\xi,\eta) \tag{23}$$

3. Method of solutions

In this section we discuss the solution methodologies of the local nonsimilarity equations governing the flow past the flat plate with variable surface temperature as well as for the plate with prescribed variable surface heat flux, namely, the series solution method, the asymptotic solution method and the implicit finite difference method respectively for small, large and all values of buoyancy and microrotation parameter, ξ.

Series solutions for smaller ξ (SRS)

To get the solutions of the present problems for small values ξ, we expand the functions, f, g, and θ in powers of ξ, substitute in the set of equations (8)-(10) for the case of prescribed surface temperature and (17)-(19) for the prescribed surface heat flux case and equate the coefficients of ξ:^{n} and get the following sets of equations:

For $n = 0$:

$$(1+\Delta)f_0''' + P_1 f_0 f_0'' - P_2 f_0'^2 + \theta_0' + \Delta g_0' = 0 \tag{24}$$

$$(1+\tfrac{\Delta}{2})g_0'' + P_1 f_0' g_0 - P_3 f_0' g_0 = 0 \tag{25}$$

$$\tfrac{1}{\Pr}\theta_0'' + P_1 f_0 \theta_0' - P_4 f_0' \theta_0 = 0 \tag{26}$$

Boundary conditions are

$$f_0(0) = f_0'(0) = 0,\ \ \theta_0(0) = 1 \text{ or } \theta'_0(0) = -1, \quad g_0(0) = -\tfrac{1}{2} f_0''(0)$$

$$f_0'(\infty) = 0,\ \ \theta_0(\infty) = g_0(\infty) = 0 \tag{27}$$

For $n \geq 1$:

$$\begin{aligned}(1+\Delta)f_n''' + \theta_n' + \Delta g_n' + \sum_{k=0}^{n}(P_1 + (n-k)P_0) f_k f_{n-k}'' \\ -(2P_2 + (n-k)P_0) f_{n-k}' f_k' = 0\end{aligned} \tag{28}$$

$$\begin{aligned}(1+\tfrac{\Delta}{2})g_n'' - \Delta(2g_{n-1} + f_{n-1}'') + \sum_{k=0}^{n}(P_1 + (n-k)P_0) f_k g_{n-k}' \\ - \sum_{k=0}^{n}(P_3 + (n-k)P_0) g_{n-k} f_k' = 0\end{aligned} \tag{29}$$

$$\tfrac{1}{\Pr}\theta_n'' + \sum_{k=0}^{n}(P_1 + (n-k)P_0) f_k \theta_{n-k}' - (P_4 + (n-k)P_0) f_{n-k}\theta_k = 0 \tag{30}$$

Corresponding boundary conditions take the form

$$f_n(0) = f_n'(0) = \theta_n(0) = 0 \text{ or } \theta_n'(0) = 0,\ g_n(0) = -\tfrac{1}{2}f_n''(0) \\ f_n'(\infty) = \theta_n(\infty) = g_n(\infty) = 0 \tag{31}$$

where

$$P_0 = \frac{1}{2}(1-n), \quad P_1 = \frac{n+3}{4}, \quad P_2 = \frac{n+1}{2},$$

$$P_3 = \frac{3n+1}{4}, \quad P_4 = n \tag{32}$$

for prescribed surface temperature and

$$P_0 = \frac{2}{5}(1-n), \quad P_1 = \frac{m+4}{5}, \quad P_2 = \frac{2m+3}{5},$$

$$P_3 = \frac{3m+2}{5}, \quad P_4 = \frac{1-m}{5} \tag{33}$$

for prescribed surface heat flux case.

Solution methodology of the above sets of equations for selected values of the parameters Pr and Δ is the sixth order implicit Runge-Kutta-Butcher method in collaboration with the Natscheim-Swigert iteration technique.

Asymptotic solutions for large ξ (ASY)

Before we get into the asymptotic solutions, we introduce the following function

$$h = g + \tfrac{1}{2}f'' \tag{34}$$

into the set of equations (8) - (10) and get

$$(1+\tfrac{1}{2}\Delta)f''' + P_1ff'' - P_2f'^2 + \theta + \Delta h' = P_0\xi\left(f'\frac{\partial f'}{\partial \xi} - f''\frac{\partial f}{\partial \xi}\right) \tag{35}$$

$$(1+\Delta)h'' + P_1fh' - P_3f'h + \frac{1}{2}\theta' - 2\Delta\xi h = P_0\xi\left(f'\frac{\partial h}{\partial \xi} - h'\frac{\partial f}{\partial \xi}\right) \tag{36}$$

$$\frac{1}{\Pr}\theta'' + P_1f\theta' - P_4f'\theta = P_0\xi\left(f'\frac{\partial \theta}{\partial \xi} - \theta'\frac{\partial f}{\partial \xi}\right) \tag{37}$$

The corresponding boundary conditions given in (12) take the form

$$f(0,\xi) = f'(0,\xi) = 0,\ \theta(0,\xi) = 1 \quad \text{or} \quad \theta'(0,\xi) = -1,\ h(0,\xi) = 0$$
$$f'(\infty,\xi) = 0,\ \theta(\infty,\xi) = g(\infty,\xi) = 0 \tag{38}$$

For the outer layer, we introduce the following

$$\begin{aligned} f(\xi,\eta) &= F_0(\eta) + \xi^{-1/2}F_1(\eta) + \quad \cdots \\ h(\xi,\eta) &= \xi^{-1}(H_0(\eta) + \xi^{-1/2}H_1(\eta) + \quad \cdots) \\ \theta(\xi,\eta) &= \Theta_0(\eta) + \xi^{-1/2}\Theta_1(\eta) + \quad \cdots \end{aligned} \tag{39}$$

Substitution of (42) into (38) - (40) yields

$$(1 + \tfrac{1}{2}\Delta)F_0''' + P_1F_0F_0'' - P_2F_0'^2 + \Theta_0 = 0 \tag{40}$$

$$\frac{1}{\mathrm{Pr}}\Theta_0'' + P_1F_0\Theta_0' - P_4F_0'\Theta_0 = 0 \tag{41}$$

$$H_0 = \frac{1}{4\Delta}\Theta_0' \tag{42}$$

$$F_0(0) = F_0'(0) = 0,\ \Theta_0(0) = 1 \quad \text{or} \quad \Theta_0'(0) = -1$$
$$F_0'(\infty) = 0, \Theta_0(\infty) = 0 \tag{43}$$

$$(1 + \tfrac{1}{2}\Delta)F_1''' + P_1F_0F_1'' - P_5F_0'F_1' + P_6F_0''F_1 + \Theta_1 = 0 \tag{44}$$

$$\frac{1}{\mathrm{Pr}}\Theta_1'' + P_1F_0\Theta_1' + P_7F_0'\Theta_1 + P_8F_1\Theta_0' = 0 \tag{45}$$

$$H_1 = \frac{1}{4\Delta}\Theta_1' \tag{46}$$

$$F_1(0) = F_1'(0) = 0,\ \Theta_1(0) = 0 \quad \text{or} \quad \Theta_1'(0) = 0$$
$$F_1'(\infty) = 0,\ \Theta_1(\infty) = 0 \tag{47}$$

Here, for non-uniform surface temperature,

$$P_5 = \frac{5n+3}{4},\ P_6 = \frac{n+1}{2},\ P_7 = \frac{1-5n}{4},\ P_8 = \frac{n+1}{2} \tag{48}$$

and for non-uniform surface heat flux

$$P_5 = \frac{5n+4}{5},\ P_6 = \frac{2m+3}{5},\ P_7 = m,\ P_8 = \frac{2m+3}{5} \tag{49}$$

For the inner layer we introduce the strained coordinate $\zeta = \xi\eta$ for prescribed surface temperature and $\zeta = \xi^{1/2\eta}$ for prescribed surface heat flux into the equations (33)-(38) and the following expressions for the functions f, h and θ:

$$f(\xi,\zeta) = \begin{cases} \xi^{-2}\big(f_0(\zeta) + \xi^{-1/2} f_1(\zeta) + \quad \cdots\big) \\ \text{for prescribed surface temperature} \\ \xi^{-1}\big(f_0(\zeta) + \xi^{-1/2} f_1(\zeta) + \quad \cdots\big) \\ \text{for prescribed surface heat flux} \end{cases} \tag{50}$$

$$h(\xi,\zeta) = h_0(\zeta) + \xi^{-1/2} h_1(\zeta) + \quad \cdots) \tag{51}$$

$$\theta(\xi,\zeta) = \begin{cases} 1 + \xi^{-1/2}\theta_0(\zeta) + \xi^{-1}\theta_1(\zeta) + \quad \cdots \\ \text{for prescribed surface temperature} \\ \xi^{-1/2}\theta_0(\zeta) + \xi^{-1}\theta_1(\zeta) + \quad \cdots \\ \text{for prescribed surface heat flux} \end{cases} \tag{52}$$

which finally leads, for the case of prescribed surface temperature, to

$$(1 + \Delta/2) f_0''' + \Delta h_0' = 0 \tag{53}$$

$$h_0'' = 0 \tag{54}$$

$$\theta_0'' = 0 \tag{55}$$

$$(1 + \Delta/2) f_1''' + \Delta h_1' = 0 \tag{56}$$

$$h_1'' = 0 \tag{57}$$

$$\theta_1'' = 0 \tag{58}$$

and for the case of prescribed surface heat flux

$$(1 + \Delta/2) f_0''' + \Delta h_0' = 0 \tag{59}$$

$$(1 + \Delta) h_0'' - 2\Delta h_0 = 0 \tag{60}$$

$$\theta_0'' = 0 \tag{61}$$

$$(1 + \Delta/2) f_1''' + \Delta h_1' = 0 \tag{62}$$

$$(1 + \Delta) h_1'' - 2\Delta h_1 = 0 \tag{63}$$

$$\theta_1'' = 0 \tag{64}$$

The boundary conditions appropriate to these equations are

$$\begin{aligned} f_0(0) = f_0'(0) = 0,\ \theta_0(0) = 1 \ \text{ or } \ \theta_0'(0) = -1\ ,\ h_0(0) = 0 \\ f_1(0) = f_1'(0) = 0,\ \theta_1(0) = 0 \ \text{ or } \ \theta_1'(0) = 0\ ,\ h_1(0) = 0 \end{aligned} \tag{65}$$

Solutions of the above equations, for the case of prescribed surface temperature, may be obtained as follows

$$f_0 = \frac{1}{2}F_0''(0)\zeta^2 - \frac{\Delta}{3(2+\Delta)}c\zeta^3 \tag{66}$$

$$h_0 = c\zeta \tag{67}$$

$$\theta_0 = \Theta_0'(0)\zeta \tag{68}$$

$$f_1 = \frac{1}{2}F_1''(0)\zeta^2 - \frac{1}{3(2+\Delta)}\zeta^3 \tag{69}$$

$$h_1 = \tfrac{1}{\Delta}\zeta \tag{70}$$

$$\theta_1 = \Theta_1'(0)\zeta \tag{71}$$

and for the surface heat flux case

$$f_0 = \frac{1}{2}F_0''(0)\zeta^2 \tag{72}$$

$$h_0 = 0 \tag{73}$$

$$\theta_0 = -\zeta \tag{74}$$

$$f_1 = \frac{1}{2}F_1''(0)\zeta^2 \tag{75}$$

$$h_1 = 0 \tag{76}$$

$$\theta_1 = 0 \tag{77}$$

The shearing stress and the rate of heat transfer coefficient now are given, for the case of prescribed surface temperature, by

$$f''(0,\xi) = \left(f_0'' + \xi^{-1/2} f_1'' + \quad \cdots\right)_{\zeta=0} \\ = \left[\left(1 + \xi^{-1/2}\right) F_0'' + \cdots\right]_{\eta=0} \tag{78}$$

$$\theta'(0,\xi) = \xi^{1/2}(\theta_0' + \xi^{-1/2}\theta_1' + \quad \cdots)_{\zeta=0} \\ = (\Theta_0' + \xi^{-1/2}\Theta_1 + \quad \cdots)_{\eta=0} \tag{79}$$

for the case of prescribed surface temperature, by

$$f''(0,\xi) = \left(f_0'' + \xi^{-1/2} f_1'' + \quad \cdots\right)_{\zeta=0} \tag{80}$$

$$\theta(0,\xi) = \xi^{-1/2}(\theta_0 + \xi^{-1/2}\theta_1 + \quad \cdots)_{\zeta=0} \tag{81}$$

In the following section an implicit finite difference is proposed to get the solutions in the entire ξ region.

Finite difference solutions (FDS)

To employ the finite difference method, the system of partial differential equations (8)-(10) with the boundary conditions (12) for the prescribed surface temperature case and differential equations (17)-(19) with the boundary conditions (20) for the prescribed surface heat flux case are first converted to a system of seven first order ordinary differential equations by introducing new unknown functions of the η derivatives. This system is then put into finite difference form in which the nonlinear difference equations are linearized by the Newton's quasi-linearization method. The resulting linear difference equations, along with the appropriate boundary conditions, are finally solved by an efficient block-tridiagonal factorization method. The details of the computational procedure have been discussed further, very recently, by Hossain *et al.* (1995, 1998). It is important to note that in initiating this method, the initial profiles at $\xi = 0$ for the functions $f(\eta)$, $g(\eta)$ and $\theta(\eta)$ and their derivative functions are incorporated from the exact solutions of the equations (24)-(26) satisfying the boundary conditions (27). Then, for a given ξ the iterative procedure is stopped to give the final values of $f(\eta), g(\eta)$ and $\theta(\eta)$ and their derivative functions in the next procedure became less than 10^{-6}, i.e. $|\delta f^i| \leq 10^{-6}$ where the superscript i denotes the number of iterations. Throughout the computations, non-uniform grids in η direction have been incorporated, considering $\eta = \sinh(j/a)$ to get quick convergence and thus save computational times and space. Finally, it should be mentioned here that the computations have been started with $\eta_\infty = 10.01$ by choosing $i_{\max} = 301$ and $a = 100$.

In the following section we discuss the results obtained from the above analyses for both the cases.

4. Results and discussion

Equations (8)-(10) with the boundary conditions (12) and equations (17)-(19) together with the boundary conditions (20) have been solved by the methodologies discussed above. The simulated results are expressed in terms of local skin-friction and the local Nusselt number only for fluid having Prandtl number, Pr = 0.72 and 7.0 while the value of the vortex-viscosity parameter Δ = 0.0, 0.1, 2.0, 5.0 and 10.0 for both the cases for variable surface heating and surface heat-flux. Effects of the aforementioned parameters are discussed in detail in the following section.

For prescribed surface temperature

For the flow from a non-isothermal plate, the numerical values of local skin friction, $Gr_x^{1/4}C_f$, and local Nusselt number, $Gr_x^{-1/4}Nu_x$, obtained by the method discussed above taking vortex viscosity parameter $\Delta = 5.0$ and temperature gradient parameter $n = 0.5$ are entered into Table I for fluids with $Pr = 0.72$ and 7.0 against the value of micro-rotation parameter ξ in the range [0, 5]. The table also compares the numerical values of local skin friction and local Nusselt number obtained by three methods, namely series solution, finite difference and asymptotic solution. From this table it may be observed that the values of local skin-friction $Gr_x^{1/4}C_f$ decrease due to an increase in the value of the Prandtl

PR	Skin-friction, $Gr_x^{1/4}C_f$				Nusselt number, $Gr_x^{-1/4}Nu_x$			
	0.72		7.0		0.72		7.0	
ξ	FDM	SRS & ASY	FDM	SRS & ASY	FDM	SRS & ASY	FDM	SRS & ASY
0.0	1.0750	1.0684[s]	0.6430	0.6346[s]	0.3655	0.3643[s]	0.6810	0.6863[s]
0.1	1.1618	1.1528	0.6988	0.7006	0.3719	0.3715	0.6945	0.7121
0.2	1.1899	1.1771	0.6996	0.7149	0.3731	0.3733	0.6961	0.7170
0.3	1.2058	1.1886	0.7129	0.7211	0.3736	0.3741	0.7003	0.7190
0.4	1.2163	1.1954	0.7130	0.7246	0.3739	0.3745	0.7012	0.7202
0.5	1.2239	1.1998	0.7175	0.7268	0.3740	0.3748	0.7025	0.7209
0.6	1.2296	1.2029	0.7196	0.7284	0.3741	0.3750	0.7078	0.7214
0.7	1.2342	1.2052	0.7252	0.7295	0.3741	0.3752	0.7097	0.7218
0.8	1.2379	1.2070	0.7275	0.7304	0.3741	0.3753	0.7099	0.7220
0.9	1.2410	1.2084	0.7284	0.7311	0.3741	0.3754	0.7102	0.7223
1.0	1.2436	1.2096	0.7304	0.7316	0.3741	0.3755	0.7111	0.7224
1.5	1.2524	1.2782	0.7473	0.7994	0.3742	0.3732	0.7144	0.7253
2.0	1.2575	.	0.7548	.	0.3742	.	0.7157	.
2.5	1.2608	.	0.7607	.	0.3742	.	0.7162	.
3.0	1.2632	.	0.7715	.	0.3742	.	0.7183	.
3.5	1.2650	.	0.7840	.	0.3742	.	0.7184	.
4.0	1.2664	.	0.7848	.	0.3743	.	0.7185	.
4.5	1.2675	.	0.7864	.	0.3744	.	0.7186	.
5.0	1.2684	1.2782[a]	0.7870	0.7994[a]	0.3744	0.3732[a]	0.7188	0.7253[a]

Notes: Here and hereafter, [s] and [a] represent the solutions due to series and asymptotic methods respectively

Table I. Numerical values of local skin friction and local Nusselt number for $\Delta = 5.0$, $n = 0.5$ for the non-uniform surface temperature with different Pr number, *Pr*; on the other hand, an increase in the value of the Prandtl number leads to an increase in the value of the local Nusselt number $Gr_x^{-1/4}Nu_x$. From this table it can also be seen that an increase in the value of the micro-rotation parameter, ξ, leads to an increase in both the skin friction $Gr_x^{1/4}C_f$ and Nusselt number $Gr_x^{-1/4}Nu_{x,}$. This behavior delays for the case of skin friction and accelerates in the case of Nusselt number as the value of the Prandtl number *Pr* is increased. The results obtained by the three methods are also found to be in excellent agreement between each other; that is, agreement of the series solutions (SRS) for small ξ and the asymptotic values (ASY) for large ξ are excellent with the corresponding values obtained from the finite difference solutions (FDM).

In Figures 2(a) and 2(b) we depict the values of the skin friction and Nusselt number respectively, for different values of the vortex-viscosity parameter Δ (= 0.1, 2.0, 5.0 and 10.0) while $Pr = 0.72$. In these figures the curves marked by solid, broken and the dotted curves represent, respectively, the results obtained by the finite difference method, series solution method and the asymptotic method. From these figures it may be seen that an increase in the value of the vortex-viscosity parameter Δ leads to an increase in the value of the local skin friction and to a decrease in the value of the local Nusselt number. We further observe that for any selected value of the vortex-viscosity parameter Δ, values of both skin friction and Nusselt number reach the respective asymptotic values smoothly. The Nusselt number reaches its asymptotic values at smaller ξ; whereas the skin friction does so at comparatively larger value of ξ. We further observe that, as the value of Δ decreases both the skin friction and Nusselt number reach their asymptotic values faster. Finally, it may be concluded that the values of the skin friction and the Nusselt number obtained by the three methods are in excellent agreement with each other when the value

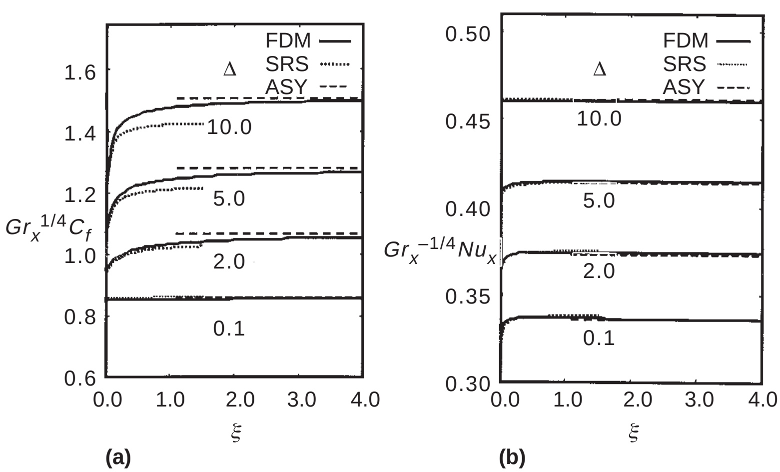

Figure 2.
Numerical values of (a) local skin-friction and (b) local Nusselt number against ξ for Δ = 0.1, 2.0, 5.0 and 10.0 while Pr = 0.72 for non-isothermal surface with n = 0.5

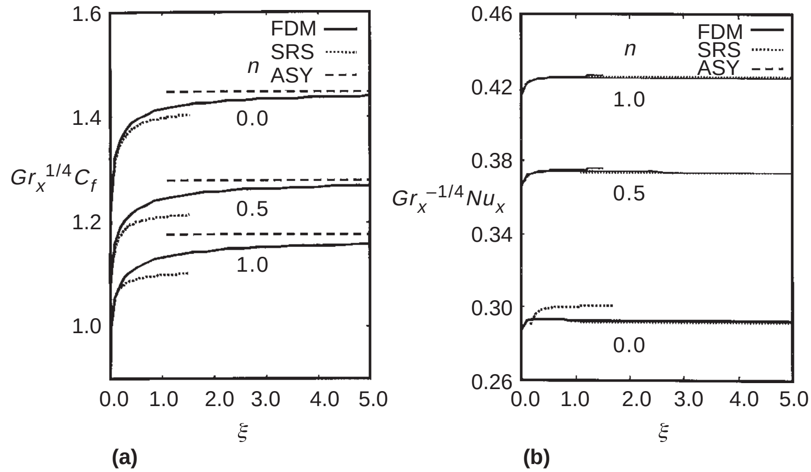

Figure 3.
Numerical values of (a) local skin-friction and (b) local Nusselt number against ξ for n = 0.0, 0.5 and 2.0 while Pr = 0.72 and Δ = 5.0

of Δ is small. On the other hand, it may also be seen that, as Δ increases the values of skin friction obtained by the series solution method and the asymptotic solution method deviate more from each other.

We now discuss the effects of dimensionless temperature gradient parameters n, on the local skin friction and the local Nusselt number. Figures 3(a) and 3(b) depict, respectively, the local skin friction and local Nusselt number at the surface of the plate for different values of n (= 0.0, 0.5 and 1.0) while Δ = 0.5 and Pr =0.72. All the results shown here are obtained from the equations valid for non-isothermal plate. It is observed from these figures that an increase in the value of n decreases the value of the skin friction while the reverse case happens for local Nusselt number. We also observe that for any selected value of n, the temperature gradient parameter, values of both the skin friction and the Nusselt number tend to their respective asymptotic values. In this case also we found the results from three methods in excellent agreement.

Prescribed surface heat-flux

The numerical values of skin friction, $Gr_x^{-3/5}C_f$, and local heat transfer rate, $Gr_x^{-1/5}Nu_x$, showing the effect of the Prandtl number Pr for the plate with non-uniform surface heat flux are entered in Table II against micro-rotation parameter ξ. As before, this table contains the values obtained by aforementioned flow regimes. It can be seen that an increase in the value of the Prandtl number leads to a decrease in the values of the skin friction and an increase in the rate of heat transfer. The effect of the micro-rotation parameter ξ for different Pr on the skin friction, $Gr_x^{-3/5}C_f$, and the rate of heat transfer, $Gr_x^{-1/5}Nu_x$, is exactly similar to the case of the flow from the non-isothermal plate.

Figures 4(a) and 4(b) represent, respectively, values for the local skin friction, $Gr_x^{-3/5}C_f$, and local Nusselt number, $Gr_x^{-1/5}Nu_x$, at the surface of the plate for Δ = 0.1, 2.0, 5.0, 10.0 while the value of the exponent of surface heat-flux m =

Table II. Numerical values of local skin friction and local Nusselt number for Δ =5.0, m = 0.5 for the non-uniform surface heat flux with different Pr

	Skin-friction, $Gr_x^{-3/5}C_f$				Nusselt number, $Gr_x^{-1/5}Nu_x$			
Pr	0.72		7.0		0.72		7.0	
ξ	FDM	SRS & ASY	FDM	SRS & ASY	FDM	SRS & ASY	FDM	SRS & ASY
0.0	1.8948	1.8892[s]	0.7747	0.7673[s]	0.4606	0.4589[s]	0.7604	0.7610[s]
0.1	1.9692	1.9911	0.8190	0.8256	0.4619	0.4663	0.7623	0.7660
0.2	1.9941	2.0292	0.8229	0.8397	0.4628	0.4663	0.7671	0.7701
0.3	2.0338	2.0490	0.8304	0.8460	0.4650	0.4663	0.7707	0.7712
0.4	2.0525	2.0613	0.8325	0.8496	0.4661	0.4664	0.7714	0.7715
0.5	2.0542	2.0695	0.8501	0.8520	0.4662	0.4664	0.7718	0.7720
0.6	2.0648	2.0755	0.8505	0.8536	0.4662	0.4664	0.7738	0.7740
0.7	2.0704	2.0800	0.8536	0.8548	0.4663	0.4664	0.7750	0.7760
0.8	2.0799	2.0836	0.8540	0.8557	0.4664	0.4665	0.7767	0.7780
0.9	2.0800	2.0864	0.8553	0.8564	0.4664	0.4665	0.7770	0.7810
1.0	2.0859	2.0887	0.8779	0.8570	0.4664	0.4665	0.7779	0.7810
1.5	2.1612	2.2237	0.9015	0.9377	0.4665	0.4665	0.7780	0.7810
2.0	2.1710	.	0.9058	.	0.4665	.	0.7784	.
2.5	2.1791	.	0.9151	.	0.4666	.	0.7785	.
3.0	2.1840	.	0.9154	.	0.4667	.	0.7786	.
3.5	2.1885	.	0.9193	.	0.4668	.	0.7787	.
4.0	2.1916	.	0.9202	.	0.4668	.	0.7788	.
4.5	2.1945	.	0.9226	.	0.4668	.	0.7788	.
5.0	2.1966	2.2237[a]	0.9233	0.9377[a]	0.4669	0.4665[a]	0.7789	0.7810[a]

0.5 for fluid having Prandtl number Pr = 0.72. From these figures it is clearly seen that the local skin friction increases and the local Nusselt number decreases as the value of Δ increases. From these figures it is further observed that both the skin friction and the Nusselt number reach their respective asymptotic values smoothly. The difference between the values of the skin friction obtained by the methods of series solution and asymptotic solution becomes significant with the increased values of Δ.

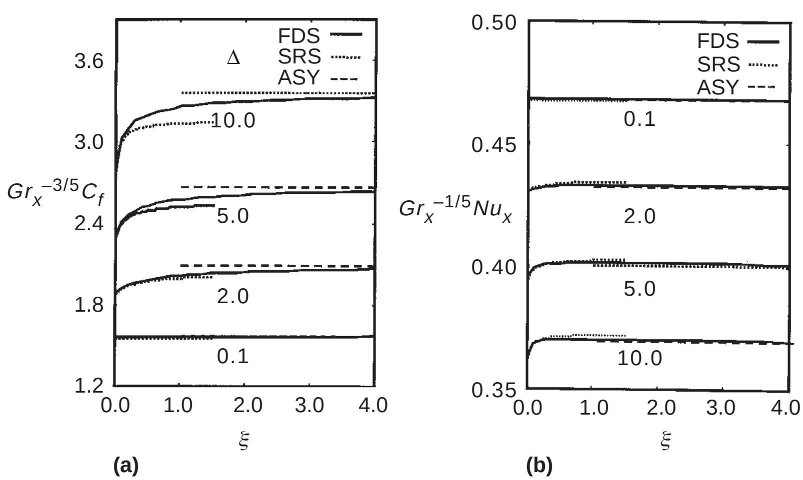

Figure 4. Numerical values of (a) local skin-friction and (b) local Nusselt number against ξ for Δ = 0.1, 2.0, 5.0 and 10.0 while Pr = 0.72 for non-isothermal surface with m = 0.5

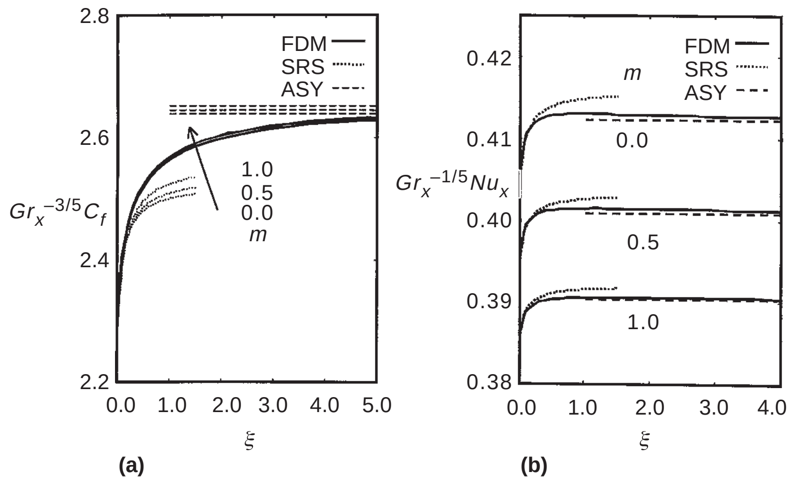

Figure 5.
Numerical values of (a) local skin-friction and (b) local Nusselt number against ξ for m = 0.0, 0.5, and 1.0 while Pr = 0.72 and Δ = 5.0

We now discuss the effects of dimensionless surface heat flux exponent, m, on the local skin-friction, $Gr_x^{-3/5}C_f$, and the local Nusselt number, $Gr_x^{-1/5}Nu_{x,}$. The numerical values of the local skin friction and local Nusselt number at the surface of the plate for different values of m (= 0.0, 0.5 and 1.0) while $\Delta = 0.5$ and $Pr = 0.72$ are depicted in Figures 5(a) and 5(b) respectively. From these figures one may easily observe that an increase in the value of m leads to an increase in the value of the local skin friction, $Gr_x^{-3/5}C_f$, and a decrease in the value of the local Nusselt number, $Gr_x^{-1/5}Nu_x$, which is just opposite to the case of non-isothermal plate. But, here, also one may observe that both the skin friction, $Gr_x^{-3/5}C_f$, and the Nusselt number, $Gr_x^{-1/5}Nu_x$, reach their respective asymptotic values as shown by the broken curves at any specific value of m.

In all the above figures the solid, dotted and broken curves represent, respectively, the finite difference solutions, perturbation series solutions and the asymptotic solutions. The comparison of the dotted and the broken curves with the solid one are found in excellent agreement. These comparisons also claim the accuracy of the finite difference solutions, since exact solutions of the equations for the cases small and large ξ were obtained, in which case the tolerance of convergence was considered as 10^{-6}.

5. Concluding remarks

In the present study we have investigated the effects of microinertia density and the vortex viscosity on laminar free convection boundary layer flow of a thermomicropolar fluid past a vertical plate with varying surface temperature as well as varying surface heat flux. The governing boundary layer equations have been simulated employing three distinct methods, namely:

(1) the series solution for small ξ (a scaled streamwise distribution of micro-inertia density);

(2) the asymptotic solution for large ξ; and

(3) the implicit finite difference method together with Keller-box scheme. Results are expressed in terms of local skin friction and local Nusselt number.

The simulated results are expressed in terms of the local skin-frictions and $Gr_x^{1/4}C_f$ and $Gr_x^{-3/5}C_f$ respectively, for prescribed surface temperature and prescribed surface heat flux case and the local Nusselt numbers $Gr_x^{-1/4}N_x$ and $Gr_x^{-1/5}N_x$, respectively, for prescribed surface temperature and prescribed surface heat flux case.

From the present investigation it may be concluded that:

- For both the non-isothermal and variable surface heat flux plates, as the value of the Prandtl number, Pr, increases, the value of the local skin friction at the surface decreases and that of the local Nusselt number increases.
- An increase in the value of the vortex viscosity parameter, Δ, leads to an increase of the local skin friction and a decrease of the local Nusselt number for both the non-isothermal and non-uniform surface heat flux cases.
- An increase in the value of the surface temperature exponent, m, leads to a decrease in the skin friction and an increase in the Nusselt number at the surface of non-isothermal plate whereas reverse effects occur on the skin friction and the Nusselt number at the surface due to an increase in value of the surface heat-flux exponent, n.

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